5. Le derivate–Parte 2

Esercizio 1. Calcolare i seguenti limiti utilizzando il Teorema di De L'Hôpital, indicando le forme indeterminate

$$\lim_{x\to 0^+} \frac{2\sin(x) - 4x}{x^2} \qquad \lim_{x\to 1} \frac{\ln(x)}{x-1} \qquad \lim_{x\to 1} \frac{x^3-1}{x^2+2x-3} \qquad \lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2}$$

$$\lim_{x\to +\infty} \frac{x^2-3x+2}{1-x^3} \qquad \lim_{x\to +\infty} \frac{e^x+5x}{x^2-3x} \qquad \lim_{x\to \frac{\pi}{2}^-} \frac{e^{\tan(x)}}{\tan(x)} \qquad \lim_{x\to -\infty} \frac{5x^3-2x^2+4}{1-2x^3}$$

$$\cancel{\lim_{x\to 0} \frac{\sin(10x)}{\sin(5x)}} \qquad \lim_{x\to \frac{\pi}{2}} \frac{\sin(x)-1}{\cos(x)} \qquad \lim_{x\to 0^+} \frac{e^{\frac{2}{x}}}{2\ln(x)} \qquad \lim_{x\to -\infty} x\cdot e^x$$

Esercizio 2. Determinare i valori dei parametri $a, b \in \mathbb{R}$ affinché le seguenti funzioni siano continue e derivabili in $\mathbb{R}$

$$f(x) = \begin{cases} a\cos^2(x) + b\sin(x), & \text{se } x < 0, \\ -\dfrac{2}{x+1}, & \text{se } x \geq 0. \end{cases} \qquad f(x) = \begin{cases} ae^x + b, & \text{se } x \leq 0, \\ \dfrac{1}{2e^x-1}, & \text{se } x > 0. \end{cases}$$

$$f(x) = \begin{cases} -2ax^2 + bx, & \text{se } x \leq 1, \\ \dfrac{1}{x^2+1}, & \text{se } x > 1. \end{cases} \qquad f(x) = \begin{cases} a + \sqrt{x^2+3}, & \text{se } x \leq 1, \\ b\ln(x) + (2a+1)x, & \text{se } x > 1. \end{cases}$$

Esercizio 3. Per ognuna delle seguentu funzioni, determinare: dominio, eventuali simmetrie, intersezioni con gli assi, segno, eventuali asintoti, massimi e minimi. Tracciare poi un grafico qualitativo.

$$f(x) = \frac{x^2}{x^2+1} \qquad f(x) = \frac{2e^x+4}{e^x-1} \qquad f(x) = \frac{e^x}{x^2+2x}$$

6. Gli integrali

Esercizio 1. Calcolare i seguenti integrali

$$\int \left(\frac{2}{\sqrt[3]{x}} + e^x\right) dx \qquad \int \frac{x+4}{x}\, dx \qquad \int \frac{x^2+4}{x^2+1}\, dx$$

$$\int (3\sin(x) + 4\sqrt{x} - \sqrt[4]{x})\, dx \qquad \int \frac{\sin(x) - 4\cos(x)}{2}\, dx \qquad \int \left(\sqrt[3]{x} - \frac{1}{x^7}\right) dx$$

$$\int (x^4 + 3x^2 - 6x - 1)\, dx \qquad \int \frac{5x+4}{x^2}\, dx \qquad \int \frac{\sqrt{x}+4}{\sqrt[3]{x}}.$$

Esercizio 2. Calcolare i seguenti integrali utilizzando l'integrazione per parti

$$\int x\cdot \ln(x)\, dx \qquad \int e^x \cdot \cos(x)\, dx \qquad \int x^2\cdot \cos(x)\, dx$$

$$\int x^2\cdot e^x\, dx \qquad \int x\cdot \arctan(x)\, dx \qquad \int x\cdot 2^x\, dx.$$

Esercizio 3. Calcolare i seguenti integrali per sostituzione usando la sostituzione indicata

$$\int \frac{1}{\sqrt[3]{x-1}}\,dx \qquad (t = x-1) \qquad \int \cos(x)\sqrt{\sin(x)}\,dx \qquad (t=\sin(x))$$

$$\int e^x \sin(e^x)\,dx \qquad (t=e^x) \qquad \int e^{5x-4}\,dx \qquad (t = 5x-4)$$

$$\int \frac{x-1}{x+1}\,dx \qquad (t = x+1) \qquad \int \frac{\ln(x)}{x}\,dx \qquad (t=\ln(x))$$

$$\int 2xe^{x^2}\,dx \qquad (t = x^2) \qquad \int \frac{4}{x-5}\,dx \qquad (t = x-5)$$

Esercizio 4. Calcolare i seguenti integrali definiti

$$\int_1^2 x\sqrt{x^2-1}\,dx \qquad \int_0^{\frac{\pi}{4}} \cos(x)\sin^3(x)\,dx \qquad \int_{-4}^{e-5} \frac{1}{x+5}\,dx$$

$$\int_1^4 \frac{1+x}{\sqrt{x}}\,dx \qquad \int_0^1 x\cdot e^{x^2}\,dx \qquad \int_0^1 x\cdot 2^x\,dx$$

Esercizio 1

1) $\lim_{x\to 0^+} \frac{2\sin(x)-4x}{x^2} = \frac{0}{0}$ F.I.

$\overset{H.}{=} \lim_{x\to 0^+} \frac{2\cos(x)-4}{2x} = \frac{2\cos(0)-4}{2\cdot 0^+} = \frac{2-4}{0^+} = \frac{-2}{0^+} = \boxed{-\infty}$

2) $\lim_{x\to 1} \frac{\ln(x)}{x-1} = \frac{\ln(1)}{1-1} = \frac{0}{0}$ F.I.

$\overset{H.}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = \boxed{1}$

3) $\lim_{x\to 1} \frac{x^3-1}{x^2+2x-3} = \frac{1-1}{1+2-3} = \frac{0}{0}$ F.I.

$\overset{H.}{=} \lim_{x\to 1} \frac{3x^2}{2x+2} = \frac{3\cdot 1}{2\cdot 1+2} = \boxed{\frac{3}{4}}$

4) $\lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2} = \frac{\sqrt{2}-\sqrt{2}}{2-2} = \frac{0}{0}$ F.I.

$\overset{H.}{=} \lim_{x\to 2} \frac{\frac{1}{2}x^{-1/2}}{1} = \lim_{x\to 2} \frac{1}{2}\frac{1}{\sqrt{x}} = \frac{1}{2}\frac{1}{\sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}}}$

5) $\lim_{x\to +\infty} \frac{x^2-3x+2}{1-x^3} = \frac{+\infty-\infty}{-\infty}$ F.I.

$\overset{H.}{=} \lim_{x\to +\infty} \frac{2x-3}{-3x^2} = \frac{+\infty}{-\infty}$ F.I.

$\overset{H.}{=} \lim_{x\to +\infty} \frac{2}{-6x} = \lim_{x\to +\infty} \frac{1}{-3x} = \frac{1}{-\infty} = \boxed{0}$

6) $\lim_{x\to +\infty} \frac{e^x+5x}{x^2-3x} = \frac{+\infty}{+\infty-\infty}$ F.I.

$\overset{H.}{=} \lim_{x\to +\infty} \frac{e^x+5}{2x-3} = \frac{+\infty}{+\infty}$ F.I.

$\overset{H.}{=} \lim_{x\to +\infty} \frac{e^x}{2} = \frac{e^{+\infty}}{2} = \boxed{+\infty}$

7) $\lim_{x\to\frac{\pi}{2}^-} \frac{e^{tg(x)}}{tg(x)} = \frac{e^{+\infty}}{+\infty} = \frac{+\infty}{+\infty}$ F.I.

$\overset{H.}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{e^{tg(x)} \cdot \cancel{\frac{1}{\cos^2(x)}}}{\cancel{\frac{1}{\cos^2(x)}}} = \lim_{x\to\frac{\pi}{2}^-} e^{tg(x)} = e^{+\infty} = \boxed{+\infty}$

8) $\lim_{x\to-\infty} \frac{5x^3-2x^2+4}{1-2x^3} = \frac{-\infty-\infty+4}{1+\infty} = \frac{-\infty}{+\infty}$ F.I.

$\overset{H.}{=} \lim_{x\to-\infty} \frac{15x^2-4x}{-6x^2} = \frac{+\infty+\infty}{-\infty} = \frac{+\infty}{-\infty}$ F.I.

$\overset{H.}{=} \lim_{x\to-\infty} \frac{30x-4}{-12x} = \frac{-\infty}{+\infty}$ F.I.

$\overset{H.}{=} \lim_{x\to-\infty} \frac{30}{-12} = -\frac{\cancel{30}^{\cancel{10}\ 5}}{\cancel{12}_{\cancel{4}\ 2}} = \boxed{-\frac{5}{2}}$

9) $\lim_{x\to\frac{\pi}{2}} \frac{\sin(x)-1}{\cos(x)} = \frac{\sin(\frac{\pi}{2})-1}{\cos(\frac{\pi}{2})} = \frac{1-1}{0} = \frac{0}{0}$ F.I.

$\overset{H.}{=} \lim_{x\to\frac{\pi}{2}} \frac{\cos(x)}{-\sin(x)} = \frac{\cos(\frac{\pi}{2})}{-\sin(\frac{\pi}{2})} = \frac{0}{-1} = \boxed{0}$

10) $\lim_{x\to0^+} \frac{e^{\frac{2}{x}}}{2\ln(x)} = \frac{e^{\frac{2}{0^+}}}{2\ln(0^+)} = \frac{e^{+\infty}}{-\infty} = \frac{+\infty}{-\infty}$ F.I.

$\overset{H.}{=} \lim_{x\to0^+} \frac{e^{\frac{2}{x}} \cdot \cancel{2} \cdot (-1)x^{-2}}{\cancel{2} \cdot \frac{1}{x}} = \lim_{x\to0^+} -\frac{e^{\frac{2}{x}} \cdot \frac{1}{x^{\cancel{2}}}}{\cancel{\frac{1}{x}}} = -\lim_{x\to0^+} e^{\frac{2}{x}} \cdot \frac{1}{x}$

$= -e^{+\infty} \cdot \frac{1}{0^+} = -(+\infty)\cdot(+\infty) = \boxed{-\infty}$

11) $\lim_{x\to-\infty} x\cdot e^x = -\infty \cdot e^{-\infty} = -\infty \cdot 0$ F.I.

ci riconduciamo ad una forma $\frac{0}{0}$ oppure $\frac{\infty}{\infty}$.

Poiché $e^x = e^{-(-x)} = (e^{-x})^{-1} = \frac{1}{e^{-x}}$, allora

$= \lim_{x\to-\infty} \frac{x}{e^{-x}} = \frac{-\infty}{+\infty}$ F.I.

$\overset{H.}{=} \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \frac{1}{-\infty} = \boxed{0.}$

Esercizio 2

1) $f(x) = \begin{cases} a\cos^2(x) + b\sin(x), & \text{se } x<0 \\ -\dfrac{2}{x+1}, & \text{se } x \geq 0 \end{cases}$

f è continua e derivabile in $\mathbb{R} \setminus \{0\}$.

Studio la continuità in $x_0 = 0$:

$$f(0) = -\frac{2}{0+1} = -2 = \lim_{x\to 0^+} f(x)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} \left(a\cos^2(x) + b\sin(x)\right) = a\cdot\cos^2(0) + b\sin(0)$$

$$= a\cdot 1 + b\cdot 0 = a$$

$\Rightarrow$ f è continua in $x_0=0$ se e solo se $\boxed{a=-2}$

Derivabilità in $x_0=0$:

- per $x>0$: $f'(x) = \left(-\dfrac{2}{x+1}\right)' = -2\,\dfrac{-1}{(x+1)^2} = \dfrac{2}{(x+1)^2}$

$$\Rightarrow \lim_{x\to 0^+} f'(x) = \frac{2}{(0+1)^2} = 2$$

- per $x<0$: $f'(x) = 2a\cos(x)\cdot(-\sin(x)) + b\cos(x)$

$$\underset{a=-2}{=} +4\sin(x)\cos(x) + b\cos(x)$$

$$\Rightarrow \lim_{x\to 0^-} f'(x) = \lim_{x\to 0} \left(4\sin(x)\cos(x) + b\cos(x)\right) = 4\cdot 0\cdot 1 + b\cdot 1 = b$$

e perciò f è derivabile in $x_0=1$ se e solo se $\boxed{b=2.}$

Soluzioni: $\boxed{a=-2,\quad b=2}$

2)
$$f(x) = \begin{cases} ae^x + b, & x \leq 0 \\ \dfrac{1}{2e^x - 1}, & x > 0 \end{cases}$$

Il dominio della seconda legge di definizione è dato da

$2e^x - 1 \neq 0 \longrightarrow e^x \neq \frac{1}{2} \longrightarrow x \neq \ln\left(\frac{1}{2}\right) \longrightarrow x \neq -\ln(2) \approx -0{,}6$

quindi f è continua e derivabile in $\mathbb{R} \setminus \{0\}$.

Continuità in $x_0 = 0$:

$f(0) = a \cdot e^0 + b = a + b = \lim_{x \to 0^-} f(x)$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \dfrac{1}{2e^x - 1} = \dfrac{1}{2-1} = 1$

$\Longrightarrow$ f continua in $x_0 = 0$ se e solo se $\boxed{a + b = 1}$

Derivabilità in $x_0 = 0$:

- per $x > 0$, $f'(x) = \left(\dfrac{1}{2e^x - 1}\right)' = \dfrac{-2e^x}{(2e^x - 1)^2}$

$\Longrightarrow \lim_{x \to 0^+} f'(x) = \dfrac{-2 \cdot e^0}{(2e^0 - 1)^2} = \dfrac{-2}{1} = -2$

- per $x < 0$, $f'(x) = ae^x \Longrightarrow \lim_{x \to 0^-} f'(x) = a \cdot e^0 = a$

$\Longrightarrow$ f derivabile in $x_0 = 0$ se e solo se $\boxed{a = -2}$

Allora,

$$\begin{cases} a + b = 1 \\ a = -2 \end{cases} \longrightarrow \begin{cases} -2 + b = 1 \\ a = -2 \end{cases} \longrightarrow \boxed{\begin{cases} b = 3 \\ a = -2 \end{cases}}$$

3) $f(x) = \begin{cases} -2ax^2+bx, & \text{se } x \leq 1 \\ \dfrac{1}{x^2+1}, & \text{se } x>1 \end{cases}$

f è continua e derivabile in $\mathbb{R}\setminus\{1\}$.

Continuità in $x_0=1$:

$$f(1) = -2a+b = \lim_{x\to 1^-} f(x)$$

$$\lim_{x\to 1^+} f(x) = \frac{1}{1+1} = \frac{1}{2}$$

$\Rightarrow$ f continua in $x_0=1$ se e solo se $\boxed{-2a+b=\frac{1}{2}}$

Derivabilità in $x_0=1$:

- se $x>1$, $f'(x) = \dfrac{-2x}{(x^2+1)^2}$ $\Rightarrow$ $\lim\limits_{x\to 1^+} f'(x) = \dfrac{-2}{(1+1)^2} = -\dfrac{1}{2}$
- se $x<1$, $f'(x) = -4ax+b$ $\Rightarrow$ $\lim\limits_{x\to 1^-} f'(x) = -4a+b$

$\Rightarrow$ f derivabile in $x_0=1$ se e solo se $\boxed{-4a+b=-\frac{1}{2}}$

Allora:

$$\begin{cases} -2a+b=\frac{1}{2} \\ -4a+b=-\frac{1}{2} \end{cases} \to \begin{cases} b=2a+\frac{1}{2} \\ -4a+2a+\frac{1}{2}=-\frac{1}{2} \end{cases} \to \begin{cases} b=2a+\frac{1}{2} \\ -2a=-1 \end{cases}$$

$$\to \boxed{\begin{cases} b=\frac{3}{2} \\ a=\frac{1}{2} \end{cases}}$$

4) $f(x) = \begin{cases} a+\sqrt{x^2+3}, & \text{se } x\le 1 \\ b\ln(x)+(2a+1)x, & \text{se } x>1 \end{cases}$

f continua e derivabile in $\mathbb{R}\setminus\{1\}$.

Continuità in $x_0=1$:

$$f(1) = a+\sqrt{1+3} = a+\sqrt{4} = a+2 = \lim_{x\to 1^-} f(x)$$

$$\lim_{x\to 1^+} f(x) = b\cdot \ln(1) + 2a+1 = 2a+1$$

$\Rightarrow$ f continua in $x_0=1$ se e solo se $a+2 = 2a+1$, cioè $\boxed{a=1}$

Derivabilità in $x_0=1$:

- per $x<1$: $f'(x) = \dfrac{x}{\sqrt{x^2+3}} \Rightarrow \lim_{x\to 1^-} f'(x) = \dfrac{1}{2}$

- per $x>1$: $f'(x) = \dfrac{b}{x} + 2a+1 \underset{a=1}{=} \dfrac{b}{x}+3$

  $\Rightarrow \lim_{x\to 1^+} f'(x) = b+3$

$\Rightarrow$ f derivabile in $x_0=1$ se e solo se $b+3 = \frac{1}{2}$, cioè $\boxed{b=-\frac{5}{2}}$

<u>Soluzione:</u> $\boxed{a=1,\ b=-\frac{5}{2}}$

Esercizio 3

1) $f(x) = \dfrac{x^2}{x^2+1}$

① DOMINIO: $x^2+1 \neq 0 \rightarrow x^2 \neq -1 \quad \forall x\in\mathbb{R} \Rightarrow \boxed{D=\mathbb{R}}$

② SIMMETRIE: $f(-x) = \dfrac{(-x)^2}{(-x)^2+1} = \dfrac{x^2}{x^2+1} = f(x) \rightarrow$ f pari

③ INTERSEZIONI ASSI:

▫ asse y: $\begin{cases} y=\dfrac{x^2}{x^2+1} \\ x=0 \end{cases} \rightarrow y=\dfrac{0}{0+1}=0 \rightarrow (0,0)$ intersezione con asse y

▫ asse x: $\begin{cases} y=\dfrac{x^2}{x^2+1} \\ y=0 \end{cases} \rightarrow \dfrac{x^2}{x^2+1}=0 \rightarrow x^2=0 \rightarrow x=0 \rightarrow (0,0)$ intersezione con asse x

④ SEGNO: $f(x)>0 \longrightarrow \dfrac{x^2}{x^2+1}>0 \quad \forall x \in \mathbb{R}\setminus\{0\}$

⑤ ASINTOTI:

- verticali: nessuno, visto che $D=\mathbb{R}$
- orizzontali: $\lim\limits_{x\to+\infty} \dfrac{x^2}{x^2+1} = \lim\limits_{x\to+\infty} \dfrac{\cancel{x^2}}{\cancel{x^2}\left(1+\frac{1}{x^2}\right)} = \dfrac{1}{1+0} = 1$

$\longrightarrow$ $y=1$ asintoto orizzontale per f se $x\to+\infty$

Essendo f pari, si ha anche

$y=1$ asintoto orizzontale per f se $x\to-\infty$

- obliqui: nessuno perché ci sono gli asintoti obliqui.

⑥ MASSIMI E MINIMI: studiamo la derivata:

$$f'(x) = \frac{2x(x^2+1)-2x\cdot x^2}{(x^2+1)^2} = \frac{\cancel{2x^3}+2x-\cancel{2x^3}}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$

Cerchiamo i punti critici:

$f'(x)=0 \longrightarrow \dfrac{2x}{(x^2+1)^2}=0 \longrightarrow 2x=0 \longrightarrow x=0$ punto critico

Studiamo ora il segno di f':

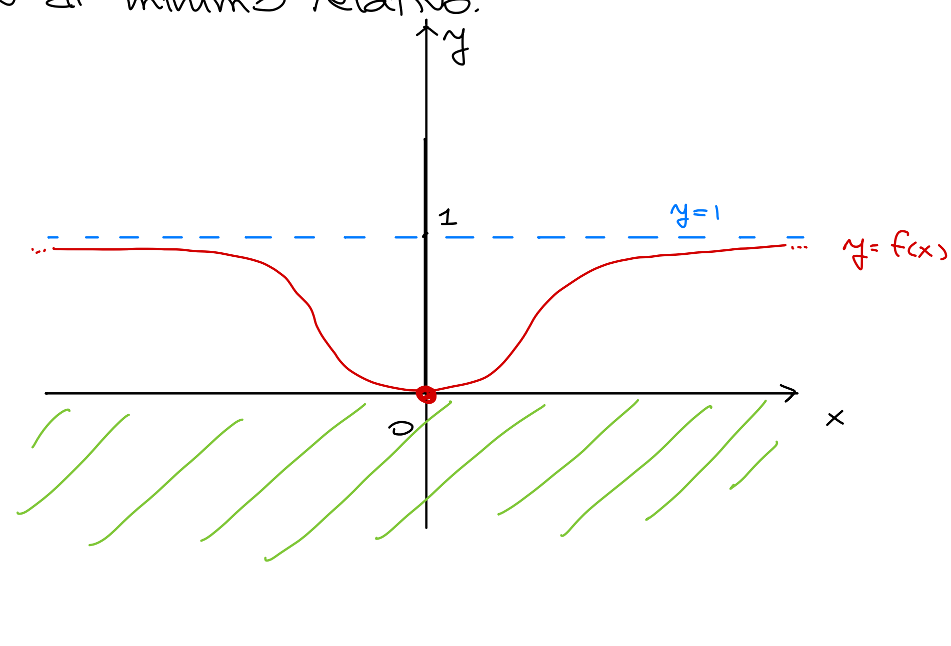
$f'(x)>0 \longrightarrow \dfrac{2x}{(x^2+1)^2}>0$

N: $x>0$

D: $(x^2+1)^2>0 \quad \forall x\in\mathbb{R}$

$\Big\}\ f'(x)>0$ se $x>0$

In $x_0=0$ f passa da essere decrescente a crescente: allora $x_0=0$ punto di minimo relativo.



2) $f(x) = \dfrac{2e^x+4}{e^x-1}$

① DOMINIO: $e^x - 1 \neq 0 \rightarrow e^x \neq 1 \rightarrow x \neq 0 \rightarrow \boxed{D = \mathbb{R} \setminus \{0\}}$

② SIMMETRIE: $f(-x) = \dfrac{2e^{-x}+4}{e^{-x}-1} \neq f(x), -f(x) \rightarrow$ f né pari, né dispari

③ INTERSEZIONI ASSI:

- asse y: nessuna, perché $0 \notin D$
- asse x: $\begin{cases} y = \dfrac{2e^x+4}{e^x-1} \\ y = 0 \end{cases} \rightarrow \dfrac{2e^x+4}{e^x-1} = 0 \rightarrow 2e^x+4=0$

$\rightarrow 2e^x = -4 \rightarrow e^x = -2 \quad \nexists x \in \mathbb{R} \rightarrow$ nessuna intersezione

④ SEGNO: $f(x) > 0 \rightarrow \dfrac{2e^x+4}{e^x-1} > 0$

N: $2e^x + 4 > 0 \rightarrow e^x > -2 \quad \forall x \in \mathbb{R}$

D: $e^x - 1 > 0 \rightarrow e^x > 1 \rightarrow x > 0$

		0	
N	+		+
D	−		+
	−		+

$\Rightarrow f(x) > 0$ se $x > 0$, $f(x) < 0$ se $x < 0$

⑤ ASINTOTI:

- verticali: $x = 0$ candidato

$$\lim_{x\to 0^+} \frac{2e^x+4}{e^x-1} = \frac{2e^{0^+}+4}{e^{0^+}-1} = \frac{2+4}{1^+-1} = \frac{6}{0^+} = +\infty$$

$$\lim_{x\to 0^-} \frac{2e^x+4}{e^x-1} = \frac{2e^{0^-}+4}{e^{0^-}-1} = \frac{2+4}{1^--1} = \frac{6}{0^-} = -\infty$$

$\Rightarrow x = 0$ asintoto verticale per f

- orizzontali:

• $\displaystyle\lim_{x\to+\infty} \frac{2e^x+4}{e^x-1} = \frac{+\infty}{+\infty}$ F.I. $\overset{H.}{=} \displaystyle\lim_{x\to+\infty} \frac{2\cancel{e^x}}{\cancel{e^x}} = \lim_{x\to+\infty} 2 = 2$

$\Rightarrow y = 2$ asintoto orizzontale per f per $x \to +\infty$

• $\displaystyle\lim_{x\to-\infty} \frac{2e^x+4}{e^x-1} = \frac{2e^{-\infty}+4}{e^{-\infty}-1} = \frac{2\cdot 0+4}{0-1} = \frac{4}{-1} = -4$

$\Rightarrow y = -4$ asintoto orizzontale per f per $x \to -\infty$.

- obliqui: nessuno, perché ci sono due asintoti orizzontali.

6 MASSIMI E MINIMI. Calcoliamo f':

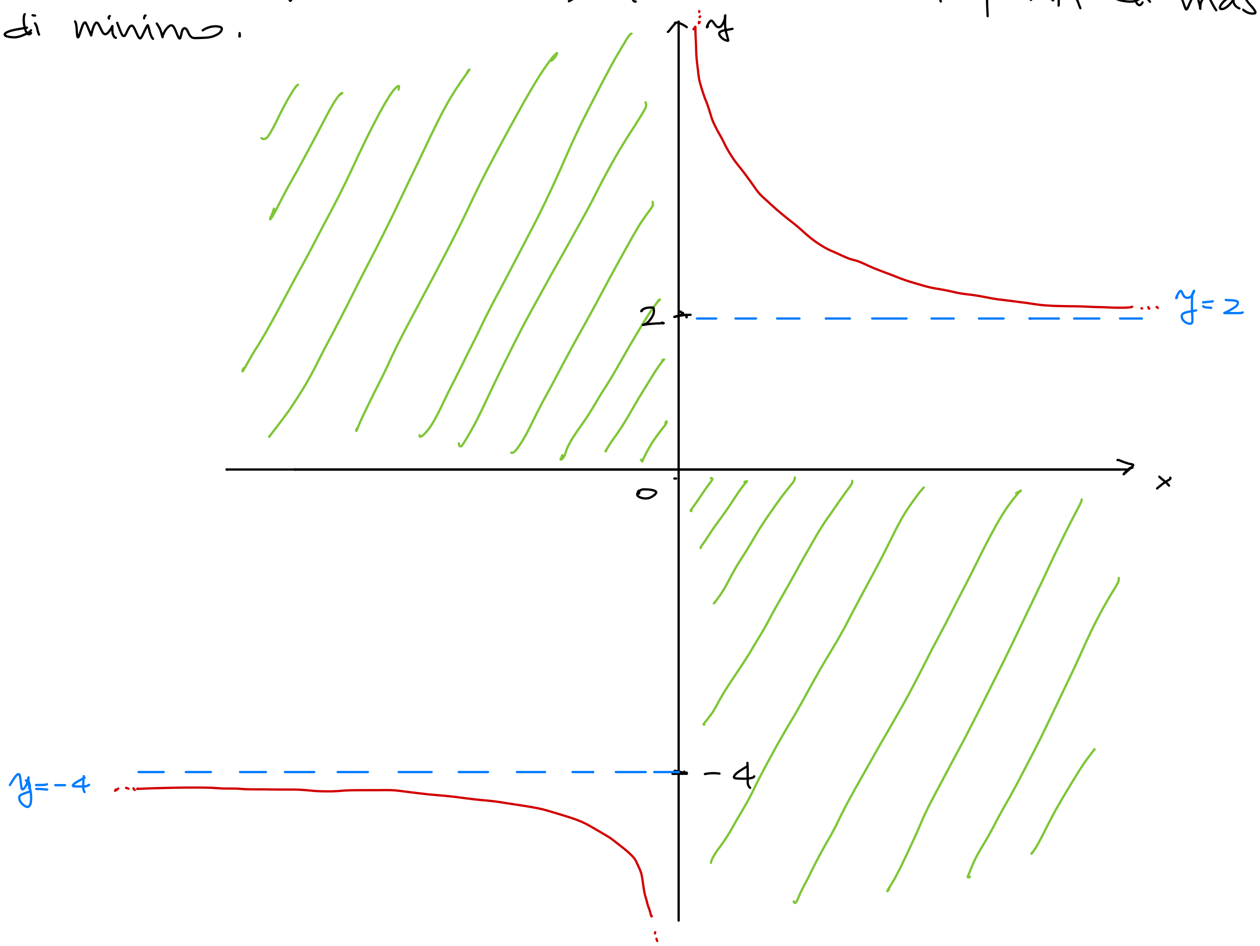
$$f'(x) = \frac{2e^x(e^x-1) - e^x(2e^x+4)}{(e^x-1)^2} = \frac{\cancel{2e^{2x}} - 2e^x \cancel{-2e^{2x}} - 4e^x}{(e^x-1)^2}$$

$e^x \cdot e^x = e^{x+x} = e^{2x}$

$$= \frac{-6e^x}{(e^x-1)^2}$$

Punti critici: $-\dfrac{6e^x}{(e^x-1)^2} = 0 \longrightarrow 6e^x = 0 \longrightarrow e^x = 0 \quad \nexists x \in \mathbb{R}$

$\Rightarrow$ non ci sono punti critici, quindi f non ha punti di massimo o di minimo.



3) $f(x) = \dfrac{e^x}{x^2+2x}$

① DOMINIO: $x^2+2x \neq 0 \rightarrow x(x+2) \neq 0 \rightarrow x \neq 0 \wedge x \neq -2$

$\Rightarrow \boxed{D = \mathbb{R} \setminus \{-2, 0\}}$

② SIMMETRIE: non essendo D simmetrico rispetto allo zero, non ha senso chiedersi se f è pari o dispari. In ogni caso, f non è né pari, né dispari.

③ INTERSEZIONI ASSI:

- asse y: nessuna, perché $0 \notin D$
- asse x: $\begin{cases} y = \dfrac{e^x}{x^2+2x} \\ y = 0 \end{cases} \rightarrow \dfrac{e^x}{x^2+2x} = 0 \rightarrow e^x = 0 \quad \nexists x \in \mathbb{R}$

$\rightarrow$ nessuna intersezione

④ SEGNO: $f(x) > 0 \rightarrow \dfrac{e^x}{x^2+2x} > 0$

N: $e^x > 0 \quad \forall x \in \mathbb{R}$

D: $x^2+2x > 0 \rightarrow x < -2 \vee x > 0$

		-2		0	
N	+		+		+
D	+		−		+
	+		−		+

$\Rightarrow f(x) > 0$ se $x < -2 \vee x > 0$, $f(x) < 0$ se $-2 < x < 0$.

⑤ ASINTOTI:

- verticali: candidati $x = -2$, $x = 0$.

$(-2^+)^2 = 4^-$

- $\lim_{x \to -2^+} \dfrac{e^x}{\underbrace{x^2+2x}} = \dfrac{e^{-2}}{0^-} = -\infty$

il denominatore è < 0 per $x > -2$

$\lim_{x \to -2^-} \dfrac{e^x}{\underbrace{x^2+2x}} = \dfrac{e^{-2}}{0^+} = +\infty$

il denominatore è > 0 per $x < -2$

$x = -2$ asintoto verticale per f

- $\lim_{x \to 0^+} \dfrac{e^x}{x^2+2x} = \dfrac{1}{0^+} = +\infty$

$\lim_{x \to 0^-} \dfrac{e^x}{x^2+2x} = \dfrac{1}{0^-} = -\infty$

$x = 0$ asintoto verticale per f

(il denominatore è > 0 per $x > 0$,
< 0 per $x < 0$)

□ orizzontali:

- $\lim_{x\to+\infty} \frac{e^x}{x^2+2x} = \frac{+\infty}{+\infty}$ F.I. $\to \lim_{x\to+\infty} \frac{e^x}{x^2+2x} = +\infty$ per gerarchia degli infiniti

$\Rightarrow$ non ci sono asintoti orizzontali per $x\to+\infty$

- $\lim_{x\to-\infty} \frac{e^x}{x^2+2x} = \frac{0}{+\infty-\infty}$ F.I. $(+\infty-\infty)$

$$= \lim_{x\to-\infty} \frac{e^x}{x^2\left(1+\frac{2}{x}\right)} = \frac{0}{+\infty(1+0)} = \frac{0}{+\infty} = 0$$

$\Rightarrow$ $y=0$ asintoto orizzontale per $x\to-\infty$

□ obliqui: (solo per $x\to+\infty$)

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{e^x}{x^2+2x}\cdot\frac{1}{x} = \lim_{x\to+\infty} \frac{e^x}{x(x^2+2x)} = \lim_{x\to+\infty} \frac{e^x}{x^3+2x^2} = \frac{\infty}{\infty} \text{ F.I.}$$

$= +\infty$ per gerarchia degli infiniti

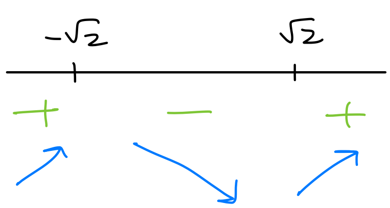
$\Rightarrow$ non ci sono asintoti obliqui

⑥ MASSIMI E MINIMI: $f'(x) = \frac{e^x(x^2+2x)-(2x+2)e^x}{(x^2+2x)^2} = \frac{e^x(x^2-2)}{(x^2+2x)^2}$

Punti critici: $f'(x)=0 \to \frac{e^x(x^2-2)}{(x^2+2x)^2} = 0 \to e^x(x^2-2)=0 \to x^2-2=0$

$\to x^2=2 \to x=-\sqrt{2} \vee x=\sqrt{2}$

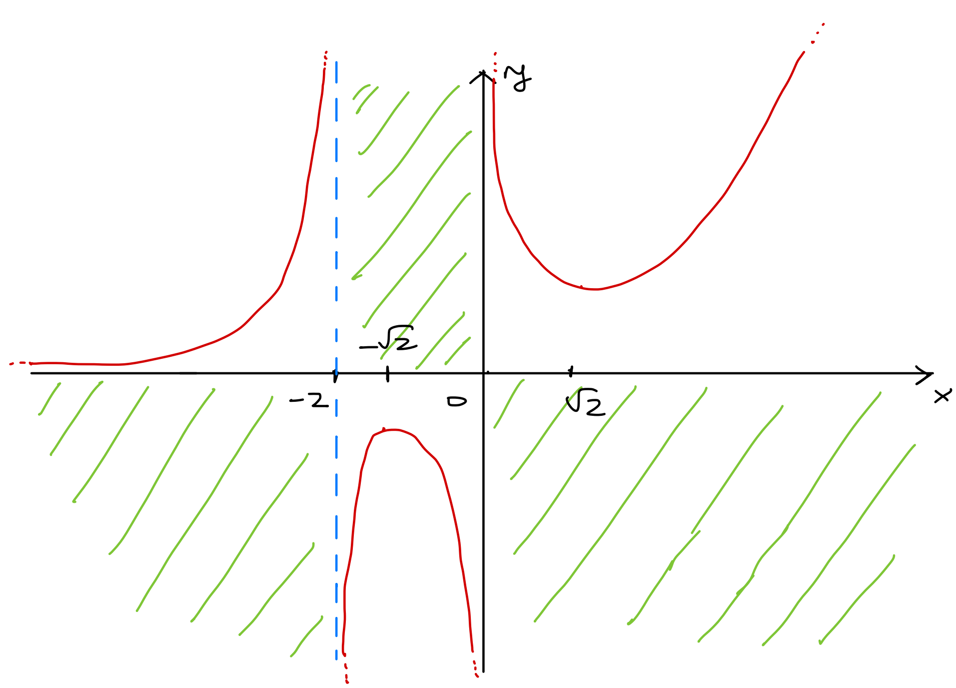
ora, $f'(x)>0 \to \frac{\overset{>0}{e^x}(x^2-2)}{\underset{>0}{(x^2+2x)^2}} > 0 \to x^2-2>0 \to x<-\sqrt{2} \vee x>\sqrt{2}$



- $x=-\sqrt{2}$ punto di massimo relativo

- $x=\sqrt{2}$ punto di minimo relativo

$\sqrt{2} \approx 1{,}414$



Esercizio 1

1) $$\int\left(\frac{2}{\sqrt[3]{x}}+e^x\right)dx = 2\int\frac{1}{\sqrt[3]{x}}dx+\int e^x dx = 2\int x^{-1/3}dx+\int e^x dx$$

$$= 2\cdot\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}+e^x+c = 2\,\frac{x^{2/3}}{\frac{2}{3}}+e^x+c$$

$$= \cancel{2}\cdot\frac{3}{\cancel{2}}\cdot x^{2/3}+e^x+c = \boxed{3\sqrt[3]{x^2}+e^x+c}$$

2) $$\int\frac{x+4}{x}dx = \int\left(\frac{x}{x}+\frac{4}{x}\right)dx = \int\left(1+\frac{4}{x}\right)dx = \int 1\cdot dx+4\int\frac{1}{x}dx$$

$$= \boxed{x+4\ln|x|+c}$$

3) $$\int\frac{x^2+4}{x^2+1}dx = \int\frac{x^2+1-1+4}{x^2+1}dx = \int\left(\frac{x^2+1}{x^2+1}+\frac{-1+4}{x^2+1}\right)dx$$

$$= \int\left(1+\frac{-3}{x^2+1}\right)dx = \int 1\cdot dx-3\int\frac{1}{x^2+1}dx$$

$$= \boxed{x-3\,\mathrm{arctg}(x)+c}$$

4) $$\int\left(3\sin(x)+4\sqrt{x}-\sqrt[4]{x}\right)dx = 3\int\sin(x)dx+4\int x^{1/2}dx-\int x^{1/4}dx$$

$$= -3\cos(x)+4\,\frac{x^{1/2+1}}{\frac{1}{2}+1}-\frac{x^{\frac{1}{4}+1}}{\frac{1}{4}+1}+c$$

$$= -3\cos(x)+4\,\frac{x^{3/2}}{\frac{3}{2}}-\frac{x^{5/4}}{\frac{5}{4}}+c$$

$$= -3\cos(x)+4\cdot\frac{2}{3}x^{3/2}-\frac{4}{5}x^{5/4}+c$$

$$= \boxed{-3\cos(x)+\frac{8}{3}\sqrt[2]{x^3}-\frac{4}{5}\sqrt[4]{x^5}+c}$$

5) $\int \frac{\sin(x) - 4\cos(x)}{2} dx = \int \frac{\sin(x)}{2} dx - \int \frac{\cancel{4}^{2}\cos(x)}{\cancel{2}} dx$

$= \frac{1}{2}\int \sin(x)\,dx - 2\int \cos(x)\,dx = \boxed{-\frac{1}{2}\cos(x) - 2\sin(x) + c}$

6) $\int \left(\sqrt[3]{x} - \frac{1}{x^7}\right) dx = \int x^{1/3} dx - \int x^{-7} dx$

$= \frac{x^{1/3+1}}{\frac{1}{3}+1} - \frac{x^{-7+1}}{-7+1} + c = \frac{x^{4/3}}{\frac{4}{3}} - \frac{x^{-6}}{-6} + c$

$= \boxed{\frac{3}{4}\sqrt[3]{x^4} + \frac{1}{6x^6} + c}$

7) $\int (x^4 + 3x^2 - 6x - 1)\,dx = \int x^4 dx + 3\int x^2 dx - 6\int x\,dx - \int dx$

$= \frac{x^5}{5} + \cancel{3}\frac{x^3}{\cancel{3}} - \cancel{6}^{3}\frac{x^2}{\cancel{2}} - x + c = \boxed{\frac{x^5}{5} + x^3 - 3x^2 - x + c}$

8) $\int \frac{5x+4}{x^2} dx = \int\left(\frac{5\cancel{x}}{x^{\cancel{2}}} + \frac{4}{x^2}\right) dx = \int\left(\frac{5}{x} + \frac{4}{x^2}\right) dx$

$= 5\int \frac{1}{x} dx + 4\int x^{-2} dx = 5\ln|x| + 4\,\frac{x^{-1}}{-1} + c$

$= \boxed{5\ln|x| - \frac{4}{x} + c}$

9) $\int \frac{\sqrt{x} + 4}{\sqrt[3]{x}} dx = \int\left(\frac{\sqrt{x}}{\sqrt[3]{x}} + \frac{4}{\sqrt[3]{x}}\right) dx = \int \frac{\sqrt{x}}{\sqrt[3]{x}} dx + 4\int x^{-1/3} dx$

$= \int \frac{x^{1/2}}{x^{1/3}} dx + 4\int x^{-1/3} dx = \int x^{\frac{1}{2}-\frac{1}{3}} dx + 4\int x^{-1/3} dx$

$\frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$

$= \int x^{1/6} dx + 4\int x^{-1/3} dx$

$= \frac{x^{\frac{7}{6}}}{\frac{7}{6}} + 4\,\frac{x^{2/3}}{\frac{2}{3}} + c = \frac{6}{7}\sqrt[6]{x^7} + \cancel{4}^{2}\cdot\frac{3}{\cancel{2}}\sqrt[3]{x^2} + c = \boxed{\frac{6}{7}\sqrt[6]{x^7} + 6\sqrt[3]{x^2} + c}$

Esercizio 2

1) $\int \underbrace{x}_{g'} \cdot \underbrace{\ln(x)}_{f}\, dx$

$f(x) = \ln(x) \longrightarrow f'(x) = \frac{1}{x}$

$g'(x) = x \longrightarrow g(x) = \frac{x^2}{2}$

$$= \frac{x^2}{2} \cdot \ln(x) - \int \frac{x^{\cancel{2}}}{2} \cdot \frac{1}{\cancel{x}}\, dx = \frac{x^2}{2} \cdot \ln(x) - \frac{1}{2}\int x\, dx$$

$$= \boxed{\frac{x^2}{2} \ln(x) - \frac{x^2}{4} + c}$$

2) $\int \underbrace{e^x}_{g'} \cdot \underbrace{\cos(x)}_{f}\, dx$

$f(x) = \cos(x) \longrightarrow f'(x) = -\sin(x)$

$g'(x) = e^x \longrightarrow g(x) = e^x$

$$= e^x \cos(x) - \int e^x \cdot (-\sin(x))\, dx = e^x \cos(x) + \int \underbrace{e^x}_{g'} \underbrace{\sin(x)}_{f}\, dx$$

$$= e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x)\, dx.$$

$f(x) = \sin(x) \to f'(x) = \cos(x)$

Dunque,

$$\int e^x \cos(x)\, dx = e^x (\cos(x) + \sin(x)) - \int e^x \cos(x)\, dx$$

$$\longrightarrow 2\int e^x \cos(x)\, dx = e^x (\cos(x) - \sin(x))$$

$$\longrightarrow \boxed{\int e^x \cos(x)\, dx = \frac{e^x (\cos(x) - \sin(x))}{2} + c}$$

3) $\int \underbrace{x^2}_{f} \underbrace{\cos(x)}_{g'}\, dx$

$f(x) = x^2 \longrightarrow f'(x) = 2x$

$g'(x) = \cos(x) \longrightarrow g(x) = \sin(x)$

$$= x^2 \sin(x) - \int 2x \cdot \sin(x)\, dx = x^2 \sin(x) - 2\int \underbrace{x}_{f} \underbrace{\sin(x)}_{g'}\, dx$$

$\longrightarrow g(x) = -\cos(x)$

$$= x^2 \sin(x) - 2\left\{ -x\cos(x) - \int 1 \cdot (-\cos(x))\, dx \right\}$$

$$= x^2 \sin(x) - 2\left\{ -x\cos(x) + \int \cos(x)\, dx \right\}$$

$$= \boxed{x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + c}$$

4) $\int \underbrace{x^2}_{f} \cdot \underbrace{e^x}_{g'} \, dx$

$f(x) = x^2 \longrightarrow f'(x) = 2x$

$g'(x) = e^x \longrightarrow g(x) = e^x$

$$= x^2 e^x - \int 2x\, e^x \, dx = x^2 e^x - 2\int \underbrace{x}_{f} \underbrace{e^x}_{g'} \, dx$$

$$= x^2 e^x - 2\left\{ x e^x - \int 1 \cdot e^x \, dx \right\}$$

$$= x^2 e^x - 2\left\{ x e^x - \int e^x \, dx \right\} = \boxed{x^2 e^x - 2x e^x + 2e^x + c}$$

5) $\int \underbrace{x}_{g'} \cdot \underbrace{\operatorname{arctg}(x)}_{f} \, dx$

$f(x) = \operatorname{arctg}(x) \longrightarrow f'(x) = \frac{1}{x^2+1}$

$g'(x) = x \longrightarrow g(x) = \frac{x^2}{2}$

$$= \frac{x^2}{2}\operatorname{arctg}(x) - \int \frac{x^2}{2} \cdot \frac{1}{x^2+1} \, dx = \frac{x^2}{2}\operatorname{arctg}(x) - \frac{1}{2}\int \frac{x^2}{x^2+1} \, dx$$

$$= \frac{x^2}{2}\operatorname{arctg}(x) - \frac{1}{2}\int \frac{x^2+1-1}{x^2+1} \, dx = \frac{x^2}{2}\operatorname{arctg}(x) - \frac{1}{2}\int\left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right) dx$$

$$= \frac{x^2}{2}\operatorname{arctg}(x) - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{x^2+1} \, dx$$

$$= \boxed{\frac{x^2}{2}\operatorname{arctg}(x) - \frac{1}{2}x + \frac{1}{2}\operatorname{arctg}(x) + c}$$

6) $\int \underbrace{x}_{f} \cdot \underbrace{2^x}_{g'} \, dx$

$f(x) = x \longrightarrow f'(x) = 1$

$g'(x) = 2^x \longrightarrow g(x) = \frac{2^x}{\ln(2)}$

$$= x \cdot \frac{2^x}{\ln(2)} - \int 1 \cdot \frac{2^x}{\ln(2)} \, dx = x \cdot \frac{2^x}{\ln(2)} - \frac{1}{\ln(2)}\int 2^x \, dx$$

$$= x\,\frac{2^x}{\ln(2)} - \frac{1}{\ln(2)} \cdot \frac{2^x}{\ln(2)} + c = \boxed{x\,\frac{2^x}{\ln(2)} - \frac{2^x}{\ln^2(2)} + c}$$

Esercizio 3

1) $\int \frac{1}{\sqrt[3]{x-1}}\,dx$

Pongo $t = x-1 \longrightarrow x = t+1 \longrightarrow dx = dt$

$$= \int \frac{1}{\sqrt[3]{t}}\,dt = \int t^{-1/3}\,dt = \frac{t^{2/3}}{\frac{2}{3}} + c = \frac{3}{2}\sqrt[3]{t^2} + c = \boxed{\frac{3}{2}\sqrt[3]{(x-1)^2} + c}$$

2) $\int \cos(x) \cdot \sqrt{\sin(x)}\,dx$

Pongo $t = \sin(x) \longrightarrow dt = (\sin(x))'\,dx = \cos(x)\,dx$

$$= \int \sqrt{t}\,dt = \int t^{1/2}\,dt = \frac{t^{3/2}}{\frac{3}{2}} + c = \boxed{\frac{2}{3}\sqrt[2]{\sin^3(x)} + c}$$

3) $\int e^x \cdot \sin(e^x)\,dx$

Pongo $t = e^x \longrightarrow x = \ln(t) \longrightarrow dx = \frac{1}{t}\,dt$

$$= \int \cancel{t} \cdot \sin(t)\,\frac{1}{\cancel{t}}\,dt = \int \sin(t)\,dt = -\cos(t) + c = \boxed{-\cos(e^x) + c}$$

4) $\int e^{5x-4}\,dx$

Pongo $t = 5x-4 \longrightarrow 5x = t+4 \longrightarrow x = \frac{1}{5}t + \frac{4}{5} \longrightarrow dx = \frac{1}{5}\,dt$

$$= \int e^t \cdot \frac{1}{5}\,dt = \frac{1}{5}\int e^t\,dt = \frac{1}{5}e^t + c = \boxed{\frac{1}{5}e^{5x-4} + c}$$

5) $\int \frac{x-1}{x+1}\,dx = \int \frac{x+1-1-1}{x+1}\,dx = \int \left(\frac{x+1}{x+1} + \frac{-1-1}{x+1}\right) dx$

$$= \int dx - 2\int \frac{1}{x+1}\,dx = x - 2\int \frac{1}{x+1}\,dx$$

Pongo $t = x+1 \longrightarrow x = t-1 \longrightarrow dx = dt$

$$= x - 2\int \frac{1}{t}\,dt = x - 2\ln|t| + c = \boxed{x - 2\ln|x+1| + c}$$

6) $\int \frac{\ln(x)}{x}\,dx$

Pongo $t=\ln(x) \longrightarrow x=e^t \longrightarrow dx=e^t\,dt$

$$=\int \frac{t}{\cancel{e^t}}\cdot \cancel{e^t}\,dt = \int t\,dt = \frac{t^2}{2}+c = \boxed{\frac{1}{2}\ln^2(x)+c}$$

7) $\int 2x\cdot e^{x^2}\,dx$

Pongo $t=x^2 \longrightarrow x=\sqrt{t} \longrightarrow dx=\frac{1}{2\sqrt{t}}\,dt$

$$=\int \cancel{2\sqrt{t}}\,e^t\cdot\frac{1}{\cancel{2\sqrt{t}}}\,dt = \int e^t\,dt = e^t+c = \boxed{e^{x^2}+c}$$

8) $\int \frac{4}{x-5}\,dx = 4\int \frac{1}{x-5}\,dx$

Pongo $t=x-5 \longrightarrow x=t+5 \longrightarrow dx=dt$

$$=4\int \frac{1}{t}\,dt = 4\ln|t|+c = \boxed{4\ln|x-5|+c}$$

Esercizio 4

1) $\int_1^2 x\sqrt{x^2-1}\,dx$

perché $1\leq x\leq 2$, quindi >0

Poniamo $t=x^2-1 \longrightarrow x^2=t+1 \longrightarrow x=\pm\sqrt{t+1} \longrightarrow x=\sqrt{t+1}$

$\longrightarrow dx=\frac{1}{2\sqrt{t+1}}\,dt$

Poi,
- $x=1 \longrightarrow t=1^2-1=0$
- $x=2 \longrightarrow t=4-1=3$

$$=\int_0^3 \cancel{\sqrt{t+1}}\cdot\sqrt{t}\cdot\frac{1}{2\cancel{\sqrt{t+1}}}\,dt = \frac{1}{2}\int_0^3\sqrt{t}\,dt = \frac{1}{2}\int_0^3 t^{1/2}\,dt$$

$$=\frac{1}{2}\left[\frac{t^{3/2}}{3/2}\right]_0^3 = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\left[t^{3/2}\right]_0^3 = \frac{1}{3}\left[3^{3/2}-0\right] = \frac{\sqrt{3^3}}{3} = \frac{\cancel{3}\sqrt{3}}{\cancel{3}} = \boxed{\sqrt{3}}$$

2) $\int_0^{\pi/4} \cos(x) \cdot \sin^3(x)\, dx$

Pongo $t = \sin(x) \longrightarrow dt = \cos(x)\, dx$

- $x = 0 \longrightarrow t = \sin(0) = 0$
- $x = \frac{\pi}{4} \longrightarrow t = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$= \int_0^{\sqrt{2}/2} t^3 dt = \frac{1}{4}\left[t^4\right]_0^{\sqrt{2}/2} = \frac{1}{4}\left[\left(\frac{\sqrt{2}}{2}\right)^4 - 0\right] = \frac{1}{\cancel{4}} \cdot \frac{\cancel{4}}{16} = \boxed{\frac{1}{16}}$$

3) $\int_{-4}^{e-5} \frac{1}{x+5}\, dx$

Pongo $t = x+5 \longrightarrow x = t-5 \longrightarrow dx = dt$

- $x = -4 \longrightarrow t = -4+5 = 1$
- $x = e-5 \longrightarrow t = e-5+5 = e$

$$= \int_1^e \frac{1}{t}\, dt = \left[\ln|t|\right]_1^e = \ln|e| - \ln|1| = \ln(e) - \ln(1) = 1 - 0 = \boxed{1}$$

4) $\int_1^4 \frac{1+x}{\sqrt{x}}\, dx = \int_1^4 \left(\frac{1}{\sqrt{x}} + \frac{x}{\sqrt{x}}\right) dx = \int_1^4 \frac{1}{\sqrt{x}}\, dx + \int_1^4 \sqrt{x}\, dx$

$$= \int_1^4 x^{-1/2} dx + \int_1^4 x^{1/2} dx = \left[\frac{x^{1/2}}{1/2}\right]_1^4 + \left[\frac{x^{3/2}}{3/2}\right]_1^4$$

$$= 2\left[x^{1/2}\right]_1^4 + \frac{2}{3}\left[x^{3/2}\right]_1^4 = 2\left(\sqrt{4} - \sqrt{1}\right) + \frac{2}{3}\left(\sqrt{4^3} - \sqrt{1^3}\right)$$

$$= 2(2-1) + \frac{2}{3}(8-1) = 2 \cdot 1 + \frac{2}{3} \cdot 7 = 2 + \frac{14}{3} = \frac{6}{3} + \frac{14}{3} = \boxed{\frac{20}{3}}$$

5) $\int_0^1 x \cdot e^{x^2} dx$

Pongo $x^2 = t \longrightarrow x = \sqrt{t} \longrightarrow dx = \frac{1}{2\sqrt{t}} dt$

- $x = 0 \longrightarrow t = 0$
- $x = 1 \longrightarrow t = 1$

$$= \int_0^1 \cancel{\sqrt{t}}\, e^t \cdot \frac{1}{2\cancel{\sqrt{t}}} dt = \frac{1}{2} \int_0^1 e^t dt = \frac{1}{2} \left[e^t \right]_0^1 = \frac{1}{2} (e - 1)$$

$$= \boxed{\frac{e-1}{2}}$$

6) $\int_0^1 \underbrace{x}_{f} \cdot \underbrace{2^x}_{g'} dx$

$f(x) = x \longrightarrow f'(x) = 1$

$g'(x) = 2^x \longrightarrow g(x) = \frac{2^x}{\ln(2)}$

$$= \left[x \cdot \frac{2^x}{\ln(2)} \right]_0^1 - \int_0^1 1 \cdot \frac{2^x}{\ln(2)} dx = 1 \cdot \frac{2}{\ln(2)} - 0 - \frac{1}{\ln(2)} \int_0^1 2^x dx$$

$$= \frac{2}{\ln(2)} - \frac{1}{\ln(2)} \cdot \left[\frac{2^x}{\ln(2)} \right]_0^1 = \frac{2}{\ln(2)} - \frac{1}{\ln(2)} \left[\frac{2}{\ln(2)} - \frac{1}{\ln(2)} \right]$$

$$= \frac{2}{\ln(2)} - \frac{1}{\ln(2)} \cdot \frac{1}{\ln(2)} = \boxed{\frac{2}{\ln(2)} - \frac{1}{\ln^2(2)}}$$